

Positive automorphisms for self-induced interval exchange transformations

Yann JULLIAN

Abstract

We give an algorithm to determine if the dynamical system generated by a positive automorphism of the free group can also be generated by a self-induced interval exchange transformation. The algorithm effectively yields the interval exchange transformation in case of success.

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Introduction

This article deals with the problem of the geometric representation of the dynamics of free group automorphisms. It is now well known ([GJLL98], [CHL09], [Cou10], [CH10]) that these can be represented by systems of partial isometries on \mathbb{R} -trees (geodesic and 0-hyperbolic metric spaces). In [CH10], the authors study fully irreducible automorphisms, and explain the representation of their dynamics depends on an attribute of the automorphism called the index ([GJLL98], [Jul10]). This index depends on the fixed subgroup and the infinite fixed points of the automorphism, and determines whether the dynamics of the automorphism is represented by a system of partial isometries on a finite union of finite (with respect to the number of points with degree at least 3) trees (this includes intervals), a finite union of non-finite trees (an example is given in [Jul11]), or a Cantor set whose convex hull is a finite (an example is given in [BK95]) or non-finite tree.

If we fix a basis A_N of the free group F_N with $N \geq 2$ generators, the free group is seen as the set of reduced words with letters in A_N or A_N^{-1} . In this article, we consider A_N -positive primitive automorphisms; an automorphism φ is A_N -positive if for any $a \in A_N$, the letters of the word $\varphi(a)$ are all in A_N , and it is primitive if there exists a positive integer k such that all the elements of A_N appear in $\varphi^k(a)$ for any $a \in A_N$. This second property ensures the minimality of the dynamic. While considering only A_N -positive automorphisms is certainly a restriction, it should be noted that the primitivity condition is weaker than the full irreducibility (a nice discussion on the subject can be found in [ABHS06]).

This article aims to answer the following question; how can we determine if an A_N -positive primitive automorphism can be represented by a system of partial isometries on an interval? We will use the early results on interval exchange transformations ([Kea75], [Kea77], [Rau79], [Vee82]). In particular, the Rauzy induction ([Rau79]) gives us a way to obtain the subshift (the combinatorial representation) of an interval exchange transformation using sequences of elementary automorphisms (called Dehn twists). We say that an interval exchange transformation is self-induced when this sequence is periodic. The main result of this paper is the following.

Theorem 0.1. *There is an algorithm able to determine in a finite number of steps if an A_N -positive primitive automorphism generates the dynamic of a self-induced interval exchange transformation.*

In case of success, the interval exchange transformation is fully determined and a Dehn twists decomposition of the automorphism is given.

The algorithm is described in great details. It is based on two main ideas. The first one is the study made in [Jul10] of the index of an A_N -positive primitive automorphism (summarized in appendix A) and the second one is a combinatorial interpretation of the Rauzy induction.

1 Automorphisms of the free group

Let F_N be the free group on $N \geq 2$ generators and let ∂F_N be its Gromov boundary. The **double boundary** $\partial^2 F_N$ is defined by

$$\partial^2 F_N = (\partial F_N \times \partial F_N) \setminus \Delta,$$

where Δ is the diagonal.

Let $A_N = \{a_0, \dots, a_{N-1}\}$ be a basis of F_N . The set of inverse letters is denoted by $A_N^{-1} = \{a_0^{-1}, \dots, a_{N-1}^{-1}\}$. Fixing A_N as a basis, we consider F_N to be the set of finite **reduced** words $v = v_0 v_1 \dots v_p$ with letters in $(A_N \cup A_N^{-1})$; reduced means for all $0 \leq i < p$, we have $v_i^{-1} \neq v_{i+1}$. The identity element of F_N will be called the empty word, and will be denoted ϵ . The set ∂F_N is the set of points $V = (V_i)_{i \in \mathbb{N}}$ with letters in $(A_N \cup A_N^{-1})$ and such that $V_i^{-1} \neq V_{i+1}$ for any $i \in \mathbb{N}$.

Remark 1.1. We will refer to elements of F_N as words, while elements of $\partial^2 F_N$ will be called points.

An automorphism φ of F_N is **A_N -positive** if for all $a \in A_N$, all the letters of $\varphi(a)$ are in A_N . When working with an A_N -positive automorphism, we always consider its representation over the free group endowed with the basis A_N . The automorphism φ induces a homeomorphism $\partial^2 \varphi$ on $\partial^2 F_N$. An A_N -positive automorphism is **primitive** if there is a positive integer k such that all letters of A_N are letters of $\varphi^k(a)$ for any $a \in A_N$.

Throughout this paper, F_N will refer to the free group endowed with the basis A_N and we always assume $N \geq 2$.

We define the shift map S on $\partial^2 F_N$:

$$\begin{aligned} S : \quad \partial^2 F_N &\rightarrow \partial^2 F_N \\ (X, Y) &\mapsto (Y_0^{-1} X, Y_0^{-1} Y), \end{aligned}$$

where Y_0 is the first letter of Y .

Let φ be an A_N -positive primitive automorphism and let a be a letter of A_N . The primitivity condition implies that we can find an integer k such that $\varphi^k(a) = pas$ where p and s are non empty words of F_N with letters in A_N . Now define

$$\begin{aligned} X &= \lim_{n \rightarrow +\infty} p^{-1} \varphi^k(p^{-1}) \varphi^{2k}(p^{-1}) \dots \varphi^{nk}(p^{-1}), \\ Y &= \lim_{n \rightarrow +\infty} as \varphi^k(s) \varphi^{2k}(s) \dots \varphi^{nk}(s). \end{aligned}$$

The **attracting subshift** of φ is the set

$$\Sigma_\varphi = \overline{\{S^n(X, Y); n \in \mathbb{Z}\}}.$$

The map S is a homeomorphism on Σ_φ . The attracting subshift only depends on φ and not on the choice of the letter a or the integer k (this is stated in different contexts in both [Que87] and [BFH97] (for example)).

2 Interval exchange transformations and completed interval exchange transformations

Let A_N be an alphabet with $N \geq 2$ letters. An **interval exchange transformation** is a bijective map f defined by:

- a pair $\pi = (\pi_0, \pi_1)$ of bijections $\pi_0, \pi_1 : A_N \rightarrow \{0, \dots, N-1\}$,
- a vector $\lambda = (\lambda_a)_{a \in A_N}$ of positive real numbers with $|\lambda| = \sum_{a \in A_N} \lambda_a$,
- for any $a \in A_N$, define $\Lambda_0(a) = \sum_{\pi_0(b) < \pi_0(a)} \lambda_b$ and $\Lambda_1(a) = \sum_{\pi_1(b) < \pi_1(a)} \lambda_b$,
and

$$\begin{aligned} f & : [0, |\lambda|) \rightarrow [0, |\lambda|) \\ x & \mapsto x - \Lambda_0(a) + \Lambda_1(a) \quad \text{if } x \in [\Lambda_0(a), \Lambda_0(a) + \lambda_a). \end{aligned}$$

From now on, an interval exchange transformation will simply be called an **IET**.

We define the **completed interval exchange transformation** δ associated to f as the system of partial isometries $\delta = \{\delta_a; a \in A_N\}$ defined by:

$$\begin{aligned} \delta_a & : [\Lambda_0(a), \Lambda_0(a) + \lambda_a] \rightarrow [\Lambda_1(a), \Lambda_1(a) + \lambda_a] \\ x & \mapsto x - \Lambda_0(a) + \Lambda_1(a). \end{aligned}$$

We say that a point x' is an **image** (resp. **pre-image**) of a point x by δ if there exists $a \in A_N$ such that $\delta_a(x) = x'$ (resp. $\delta_a^{-1}(x) = x'$). It is important to notice that the isometries are defined on closed sets, effectively resulting in points with more than one image (resp. pre-image). From now on, a completed interval exchange transformation will simply be called a **CIET**.

For convenience, we define $\delta_{a^{-1}} = \delta_a^{-1}$ for any $a \in A_N$. For any point $x \in [0, |\lambda|]$, define

$$\Sigma_\delta(x) = \{(U, V) \in \partial^2 F_N; \forall n \in \mathbb{N}, \quad \begin{aligned} & U_n^{-1} \in A_N \text{ and } x \in D(\delta_{U_n} \circ \dots \circ \delta_{U_0}) \\ & V_n \in A_N \text{ and } x \in D(\delta_{V_n} \circ \dots \circ \delta_{V_0}) \end{aligned}\}.$$

where D denotes the domain. The **δ -subshift** is the S -invariant set $\Sigma_\delta = \bigcup_{x \in [0, |\lambda|]} \Sigma_\delta(x)$. The points $\Lambda_0(a)$ for $a \neq \pi_0^{-1}(0)$ are the **forward δ -singularities** and the points $\Lambda_1(a)$ for $a \neq \pi_1^{-1}(0)$ are the **backward δ -singularities**.

From now on, defining an IET $f = (\pi, \lambda)$ and its associated CIET δ implicitly defines the maps $\pi_0, \pi_1 : A_N \rightarrow \{0, \dots, N-1\}$, the points $\Lambda_0(a)$ and $\Lambda_1(a)$ for any $a \in A_N$, the sets $\Sigma_\delta(x)$ for any $x \in [0, |\lambda|]$ and the δ -subshift Σ_δ .

2.1 The Keane condition

An IET $f = (\pi, \lambda)$ is **minimal** if there is no closed f -invariant subset of $[0, |\lambda|)$. An IET $f = (\pi, \lambda)$ satisfies the Keane condition (see [Kea75], [Kea77]) if for any $n \in \mathbb{N}$ and any $a, b \neq \pi_0^{-1}(0)$, we have $f^n(\Lambda_0(a)) \neq \Lambda_0(b)$ (there are no connections). It was proven in [Kea75] that this condition suffice to ensure minimality. We study the effect of this condition on the associated CIET.

Proposition 2.1. *Let $f = (\pi, \lambda)$ be an IET satisfying the Keane condition and let δ be the CIET associated to f . Then for any $x \in [0, |\lambda|]$, the set $\Sigma_\delta(x)$ contains at most two points.*

Proof. First, observe that the Keane condition implies that if x is a forward (resp. backward) δ -singularity, then for any positive interger n , the point $f^{-n}(x)$ (resp. $f^n(x)$) can not be a backward (resp. forward) δ -singularity.

It is obvious that if $\Sigma_\delta(x)$ contains more than one point, then there exist a point $(U, V) \in \Sigma_\delta(x)$ and an integer n such that either $\delta_{V_n} \circ \dots \circ \delta_{V_0}(x)$ or $\delta_{U_n} \circ \dots \circ \delta_{U_0}(x)$ is a δ -singularity. We directly assume x is a δ -singularity and we suppose $\Sigma_\delta(x)$ contains more than two points. Suppose it is a forward δ -singularity (the proof for backward δ -singularities is similar). There are two cases. In the first case, there exist $(U, V) \in \Sigma_\delta(x)$ and an integer m such that $\delta_{U_m} \circ \dots \circ \delta_{U_0}(x) = x'$ is a backward δ -singularity; assuming m is the smallest such integer, we obtain $f^{-(m+1)}(x) = x'$, which contradicts the Keane condition. In the second case, there exist $(U, V) \in \Sigma_\delta(x)$ and an integer m such that $\delta_{V_m} \circ \dots \circ \delta_{V_0}(x) = x''$ is a forward δ -singularity. Note that either $\delta_{V_0}(x)$ or $\delta_{V_1} \circ \delta_{V_0}(x)$ is a backward δ -singularity y (in the latter case, we have $\delta_{V_0}(x) = |\lambda|$). Assuming again m is the smallest integer such that $\delta_{V_m} \circ \dots \circ \delta_{V_0}(x)$ is a forward δ -singularity, we obtain $f^m(y) = x''$ (resp. $f^{m-1}(y) = x''$) if $\delta_{V_0}(x) = y$ (resp. $\delta_{V_1} \circ \delta_{V_0}(x) = y$), a contradiction. \square

Corollary 2.2. *Let x be a point with $\Sigma_\delta(x) = \{(U, V), (U', V')\}$. Then we have either $U = U'$ or $V = V'$.*

We now state an important result of the present article. This theorem gives a simple definition of the δ -subshift of a CIET δ associated to an IET f satisfying the Keane condition. The result can be found in [Kea75]; we give an alternative proof.

We deduce from proposition 2.1 that if f satisfies the Keane condition, then the set $\Sigma_\delta(0)$ contains exactly one point.

Theorem 2.3 ([Kea75]). *Let $f = (\pi, \lambda)$ be an IET satisfying the Keane condition and let δ be the CIET associated to f . Assume $\Sigma_\delta(0) = \{Z\}$. Then*

$$\overline{\{S^n(Z); n \in \mathbb{Z}\}} = \Sigma_\delta.$$

Proof. Let $W = (U, V)$ be a point of $\overline{\{S^n(Z); n \in \mathbb{Z}\}}$. For any $n \in \mathbb{N}$, define $D_n = D(\delta_{V_n} \circ \dots \circ \delta_{V_0}) \cap D(\delta_{U_n} \circ \dots \circ \delta_{U_0})$ (where D denotes the domain); note that D_n is a closed interval and that $D_{n+1} \subset D_n$. Moreover, for any $n \in \mathbb{N}$, there exists a point $W' = (U', V') \in \{S^p(Z); p \in \mathbb{Z}\}$ such that, for any $0 \leq i \leq n$,

we have $V'_i = V_i$ and $U'_i = U_i$, meaning D_n is not empty. We conclude $\bigcap_{n \in \mathbb{N}} D_n$ contains (at least) one point x with $\Sigma_\delta(x) = W$, and $\overline{\{S^n(Z); n \in \mathbb{Z}\}} \subset \Sigma_\delta$.

Let $W = (U, V)$ be a point of Σ_δ . We define again $D_n = D(\delta_{V_n} \circ \dots \circ \delta_{V_0}) \cap D(\delta_{U_n} \circ \dots \circ \delta_{U_0})$ and we prove the following lemma.

Lemma 2.4. *For any $n \in \mathbb{N}$, the set D_n is a non trivial (not a singleton) closed interval.*

Proof. It is obvious that D_n is a non empty (since $W \in \Sigma_\delta$) closed interval for any $n \in \mathbb{N}$. Suppose D_n is trivial for some n . We may simply assume there exists an integer $m \geq 2$ such that $D(\delta_{V_m} \circ \dots \circ \delta_{V_0}) = \{x\}$ is trivial (if it is not the case, we can work with $S^k(W)$ for some $k \in \mathbb{Z}$). Also assume m is minimal in the sense that neither $D(\delta_{V_m} \circ \dots \circ \delta_{V_1})$ nor $D(\delta_{V_{m-1}} \circ \dots \circ \delta_{V_0})$ is trivial.

In that case, x is either 0 or $|\lambda|$ or a forward δ -singularity. If it were not, we could choose a positive real number τ such that $(x - \tau, x + \tau) \subset D(\delta_{V_0})$, and since $D(\delta_{V_m} \circ \dots \circ \delta_{V_1})$ is not trivial and contains $\delta_{V_0}(x)$, the set $D(\delta_{V_m} \circ \dots \circ \delta_{V_0})$ would not be trivial.

Similarly, the point $\delta_{V_{m-1}} \circ \dots \circ \delta_{V_0}(x)$ must also be either 0 or $|\lambda|$ or a forward δ -singularity. If it were not, we could again choose τ so that $(y - \tau, y + \tau) \subset D(\delta_{V_m})$ with $y = \delta_{V_{m-1}} \circ \dots \circ \delta_{V_0}(x)$, resulting in a non trivial set $D(\delta_{V_m} \circ \dots \circ \delta_{V_0})$.

This contradicts proposition 2.1. \square

Recall from [Kea75], that since f satisfies the Keane condition, f is minimal. In particular, for any $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $f^{k_n}(0) \in D_n$. We choose $k_n \geq n + 1$ to avoid problems related to the fact that $f(0)$ is a backward δ -singularity. Define $S^{k_n}(Z) = W^{(n)}$ for any $n \in \mathbb{N}$, observe $(W^{(n)})_n$ converges to W and conclude. \square

From now on, we only work with CIETs. A CIET δ will simply be defined as a pair (π, λ) , and following from proposition 2.1, we say δ verifies the Keane condition if $\Sigma_\delta(x)$ contains at most two points for any $x \in [0, |\lambda|]$.

2.2 Rauzy induction and positive automorphisms

The following constructions are given (for IETs) in [Rau79]. Let $\delta = (\pi, \lambda)$ be a CIET. Define $\alpha_0 = \pi_0^{-1}(N - 1)$ and $\alpha_1 = \pi_1^{-1}(N - 1)$ and assume $\lambda_{\alpha_0} \neq \lambda_{\alpha_1}$ (note this is automatically true if δ satisfies the Keane condition).

Suppose $\lambda_{\alpha_0} > \lambda_{\alpha_1}$. We say δ has **type 0**. The **completed Rauzy induction** of δ is the CIET $\delta' = (\pi', \lambda')$ defined by

- $\pi'_0 = \pi_0$,
 - $\forall a \in A_N; \pi_1(a) \leq \pi_1(\alpha_0), \pi'_1(a) = \pi_1(a)$,
 - $\pi'_1(\alpha_1) = \pi_1(\alpha_0) + 1$,
 - $\forall a \in A_N; \pi_1(\alpha_0) < \pi_1(a) < \pi_1(\alpha_1), \pi'_1(a) = \pi_1(a) + 1$.
- $\lambda'_{\alpha_0} = \lambda_{\alpha_0} - \lambda_{\alpha_1}$ and $\forall a \neq \alpha_0, \lambda'_a = \lambda_a$.

If we simply name I'_a the domain of δ'_a for any $a \in A_N$, then $\delta'_{\alpha_1}(I'_{\alpha_1}) = \delta_{\alpha_0} \circ \delta_{\alpha_1}(I'_{\alpha_1})$ and $\delta'_a(I'_a) = \delta_a(I'_a)$ for any $a \neq \alpha_1$.

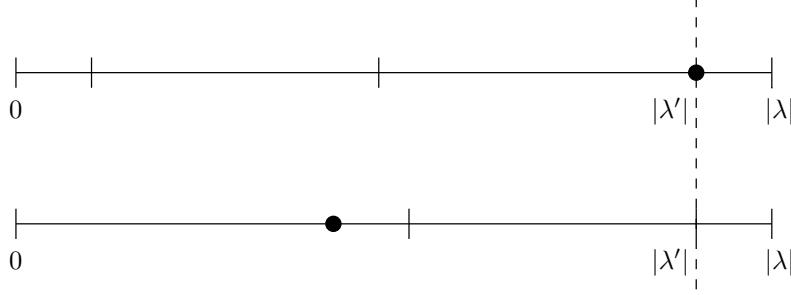


Figure 1: An example of type 0 induction.

If $\lambda_{\alpha_0} < \lambda_{\alpha_1}$, then we say δ has **type 1**. The **completed Rauzy induction** of δ is the CIET $\delta' = (\pi', \lambda')$ defined by

- $\pi'_1 = \pi_1$,
 - $\forall a \in A_N; \pi_0(a) \leq \pi_0(\alpha_1), \pi'_0(a) = \pi_0(a)$,
 - $\pi'_0(\alpha_0) = \pi_0(\alpha_1) + 1$,
 - $\forall a \in A_N; \pi_0(\alpha_1) < \pi_0(a) < \pi_0(\alpha_0), \pi'_0(a) = \pi_0(a) + 1$.
- $\lambda'_{\alpha_1} = \lambda_{\alpha_1} - \lambda_{\alpha_0}$ and $\forall a \neq \alpha_1, \lambda'_a = \lambda_a$.

Simply naming I'_a the domain of δ'_a for any $a \in A_N$, we have $\delta'_{\alpha_0}(I'_{\alpha_0}) = \delta_{\alpha_0} \circ \delta_{\alpha_1}(I'_{\alpha_0})$ and $\delta'_a(I'_a) = \delta_a(I'_a)$ for any $a \neq \alpha_0$.

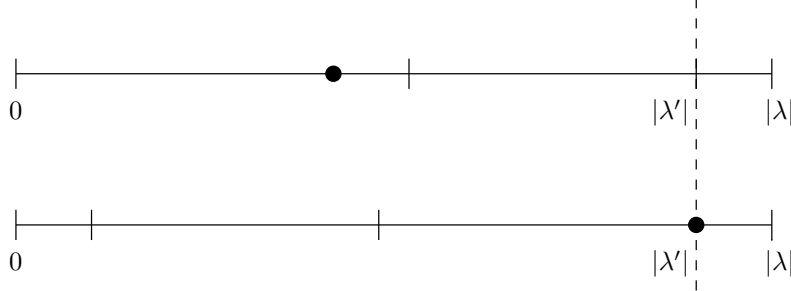


Figure 2: An example of type 1 induction.

We write $\delta' = R(\delta)$ in both cases.

Remark 2.5. Observe that the induction can be seen as a first return system on $[0, |\lambda'|]$ except for one point: the left end point of I'_{α_1} (resp. I'_{α_0}) if δ has type 0 (resp. type 1).

2.2.1 Rauzy classes

A pair $\pi = (\pi_0, \pi_1)$ is called **reducible** if $\pi_1 \circ \pi_0^{-1}(\{0, \dots, k\}) = \{0, \dots, k\}$ for some $k < N - 1$. Irreducibility is a consequence of the Keane condition, and in fact, we are only interested in irreducible pairs.

Given irreducible pairs $\pi = (\pi_0, \pi_1)$ and $\pi' = (\pi'_0, \pi'_1)$ where $\pi_0, \pi_1, \pi'_0, \pi'_1 : A_N \rightarrow \{0, \dots, N - 1\}$ are bijections, we say that π' is a successor of π if there exist two vectors λ, λ' of \mathbb{R}^N with positive entries such that $R(\pi, \lambda) = (\pi', \lambda')$. Any pair π has exactly two successors, corresponding to types 0 and 1, and any pair π' is the successor of exactly two pairs. This relation defines a partial order on the sets of irreducible pairs that may be represented by a directed graph called the **Rauzy graph**; obviously, there is one graph per value of N . The connected components of such graphs are called **Rauzy classes**. An exemple is given on figure 3.

Observe that if $\delta = (\pi, \lambda)$ is a CIET satisfying the Keane condition, then for any $n \in \mathbb{N}$, the map $R^n(\delta)$ is defined. Hence, δ determines an infinite path in one of the Rauzy classes. Moreover, each edge of the Rauzy graph comes with an elementary automorphism (a Dehn twist) intended to specify the transition from one subshift to the other (see figure 3). Define $R(\delta) = \delta' = (\pi', \lambda')$ and $\alpha_0 = \pi_0^{-1}(N - 1), \alpha_1 = \pi_1^{-1}(N - 1)$. If δ has type 0, define the automorphism σ of F_N by

$$\begin{aligned} \sigma & : \quad \alpha_1 \mapsto \alpha_1 \alpha_0 \\ & \quad a \mapsto a \quad \text{if } a \neq \alpha_1. \end{aligned}$$

If δ has type 1, define the automorphism σ of F_N by

$$\begin{aligned} \sigma & : \quad \alpha_0 \mapsto \alpha_1 \alpha_0 \\ & \quad a \mapsto a \quad \text{if } a \neq \alpha_0. \end{aligned}$$

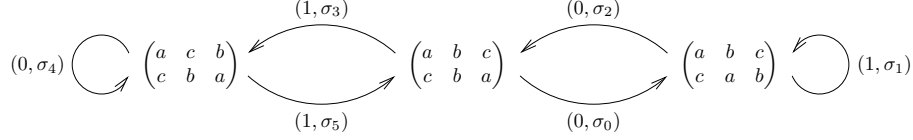
We obtain the following proposition.

Proposition 2.6. *For any point $x \in [0, |\lambda'|]$, we have $\Sigma_\delta(x) = \{\partial^2 \sigma(W); W \in \Sigma_{\delta'}(x)\}$. The set $\Sigma_{\delta'}(|\lambda'|)$ contains only one point W and $\partial^2 \sigma(W) \in \Sigma_\delta(|\lambda'|)$.*

Note that $\Sigma_\delta(|\lambda'|)$ contains two points. However, the point $|\lambda'|$ is the only point of $[0, |\lambda'|]$ such that $\Sigma_\delta(|\lambda'|)$ contains a point (U, V) with $U_0 = \alpha_1^{-1}$ and $V_0 = \alpha_0$. This means $(U', V') = \partial^2 \sigma^{-1}(U, V)$ can not be a point of $\Sigma_{\delta'}$; in particular the first letter of U' (resp. V') if δ has type 0 (resp. type 1) is in A_N (resp. A_N^{-1}).

It is also important to understand how singularities are created. Namely, if S_δ (resp. $S_{\delta'}$) is the set of δ -singularities (resp. δ' -singularities) then $S_{\delta'} \setminus S_\delta$ contains exactly one point x . This point only has one image and one pre-image by δ ; however, if δ has type 0 (resp. type 1), then its pre-image x'' (resp. image x') is a backward (resp. forward) δ -singularity, meaning $\Sigma_\delta(x) = \{(U, V), (U', V')\}$ contains two points. Observe that we have $U_0 U_1 = \alpha_0^{-1} \alpha_1^{-1}$ and $U'_0 U'_1 = \alpha_0^{-1} \beta_0^{-1}$ with $\pi_1(\beta_0) = N - 2$ (resp. $V_0 V_1 = \alpha_1 \alpha_0$ and $V'_0 V'_1 = \alpha_1 \beta_1$ with $\pi_0(\beta_1) = N - 2$) if δ has type 0 (resp. type 1), and conclude $\Sigma_{\delta'}(x) = \{\partial^2 \sigma^{-1}(U, V), \partial^2 \sigma^{-1}(U', V')\}$ effectively is a δ' -singularity.

Proposition 2.7. *If δ has type 0 (resp. type 1), then the pre-image (resp. image) by δ of the left end point of the domain of $\delta'_{\alpha_1^{-1}}$ (resp. δ'_{α_0}) is the left end point of the domain of $\delta_{\alpha_1^{-1}}$ (resp. δ_{α_0}).*



Each edge is labeled with a pair (τ, σ) representing the type and the associated recoding automorphism; these automorphisms are defined by:

$$\begin{array}{lll}
 \sigma_0 : & a \mapsto ac & \sigma_1 : & a \mapsto a & \sigma_2 : & a \mapsto a \\
 & b \mapsto b & & b \mapsto b & & b \mapsto bc \\
 & c \mapsto c & & c \mapsto bc & & c \mapsto c \\
 \\
 \sigma_3 : & a \mapsto a & \sigma_4 : & a \mapsto ab & \sigma_5 : & a \mapsto a \\
 & b \mapsto b & & b \mapsto b & & b \mapsto ab \\
 & c \mapsto ac & & c \mapsto c & & c \mapsto c
 \end{array}$$

Figure 3: An example of Rauzy class.

2.2.2 Self-induced interval exchange transformation

We say that a CIET $\delta^{(0)} = (\pi^{(0)}, \lambda^{(0)})$ is **self-induced** if there exists a positive integer n such that $R^n(\delta^{(0)}) = (\pi^{(n)}, \lambda^{(n)})$ with

- $\pi^{(n)} = \pi^{(0)}$,
- $\lambda^{(n)} = \eta \lambda^{(0)}$ for some positive real number η .

Alternatively, one can say a CIET is self-induced if its associated path in the Rauzy graph is periodic. The path $(\pi^{(0)}, \pi^{(1)}, \dots, \pi^{(n)} = \pi^{(0)})$ is a cycle of the Rauzy graph. Assume it is minimal: it is not a power of a smaller cycle. Suppose σ_i is the automorphism associated (as in the previous section) to the edge $(\pi^{(i)}, \pi^{(i+1)})$ for any $0 \leq i < n$.

We define the **δ -automorphism** φ as the automorphism $\varphi = \sigma_0 \circ \dots \circ \sigma_{n-1}$. The automorphism φ is obviously A_N -positive. Moreover, we get from [Yoc06, corollary 3] that δ satisfies the Keane condition, and from [Yoc06, corollary 4] that φ is primitive.

Theorem 2.8. *Let $\delta = (\pi, \lambda)$ be a self-induced CIET and let φ be the associated δ -automorphism. Define the δ -subshift Σ_δ and the attracting subshift Σ_φ of φ . We have $\Sigma_\delta = \Sigma_\varphi$.*

Proof. Recall that since δ satisfies the Keane condition, the set $\Sigma_\delta(0)$ contains exactly one point Z . Thanks to theorem 2.3, we only need to prove Z is in Σ_φ .

Let n be the smallest positive integer such that $R^n(\delta) = \delta' = (\pi', \lambda')$ with $\pi' = \pi$ and $\lambda' = \eta\lambda$ ($\eta > 0$). Observe that $\Sigma_\delta(0) = \Sigma_{\delta'}(0) = \{Z\}$. We deduce from proposition 2.6 that $\partial^2\varphi(Z) = Z$. Define $Z = (X, Y)$. Considering any letter Y_n (resp. X_n^{-1}) is an element of A_N and since φ is A_N -positive primitive, it is easy to see that $Y = \lim_{n \rightarrow +\infty} \varphi^n(Y_0)$ and $X = \lim_{n \rightarrow +\infty} \varphi^n(X_0)$. To complete the proof, we also need to prove that the word $X_0^{-1}Y_0$ is contained in $\varphi^k(a)$ for some $k \geq 1$ and $a \in A_N$. By definition, $Y_0 = \pi_0^{-1}(0)$ and $X_0^{-1} = \pi_1^{-1}(0)$. Observe $D(\delta_{Y_0}) \cap D(\delta_{X_0})$ (where D denotes the domain) is not trivial. By minimality, there exists a point x in $D(\delta_{Y_0}) \cap D(\delta_{X_0})$ and a positive integer n such that $S^n(Z) \in \Sigma_\delta(x)$, and we deduce the word $X_0^{-1}Y_0$ is contained in $\varphi^k(Y_0)$ for some k . We conclude Z is a point of Σ_φ and $\Sigma_\delta = \Sigma_\varphi$. \square

This definition of self-induction is somewhat non canonical because the Rauzy induction always happens on the right. Consider a self-induced CIET $\delta = (\pi, \lambda)$. Define the CIET $\delta' = (\pi', \lambda)$ such that, for any $a \in A_N$, we have $\pi'_0(a) = N - 1 - \pi_0(a)$ and $\pi'_1(a) = N - 1 - \pi_1(a)$. The CIETs δ and δ' are essentially the same (if δ is the CIET on the interval $[0, |\lambda|]$, then δ' can be seen as the same CIET on $[|\lambda|, 0]$) and we have $\Sigma_{\delta'} = \Sigma_\delta$. However, δ' may not be auto-induced in our sense.

For example, let η be the greatest root of $\eta^2 - 4\eta + 1$. Define $\lambda_a = 2\eta - 1$, $\lambda_b = \eta$ and $\lambda_c = 2\eta$. Also define the permutations

$$\pi = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} \quad \text{and} \quad \pi' = \begin{pmatrix} c & b & a \\ b & a & c \end{pmatrix}$$

and the CIETs $\delta = (\pi, \lambda)$ and $\delta' = (\pi', \lambda)$. We obtain $R^5(\delta) = (\pi, \eta^{-1}\lambda)$, so δ is effectively self-induced. However, for any positive integer n , there will never be a real number η' such that $R^n(\delta')$ is the CIET $(\pi', \eta'\lambda)$: a simple iteration shows that we obtain $R^5(\delta') = (\pi'', \lambda'')$ with

$$\pi'' = \begin{pmatrix} c & a & b \\ b & a & c \end{pmatrix} \quad \text{and} \quad \lambda'' = \begin{pmatrix} 1 \\ \eta - 2 \\ 1 \end{pmatrix}$$

and $R^5(\pi'', \lambda'') = (\pi'', \eta^{-1}\lambda'')$.

It is reasonable to think this is the general behavior of self-induced CIETs. Namely, if δ is a self-induced CIET and we define δ' as above, then the path of the Rauzy graph associated to δ' is eventually periodic. The issue is addressed in proposition 3.3.

3 The algorithm

Theorem 2.8 is especially useful because it allows us to translate obvious geometric properties into combinatorial properties. The important point here is,

given a CIET δ satisfying the Keane condition, the attracting subshift of the δ -automorphism must contain pairs of points representing the coding of the orbits of the δ -singularities. This is the main idea of the algorithm below.

Following from [Jul10], we give a combinatorial definition of singularities. For any $w \in F_N$, the conjugacy i_w is the automorphism of F_N defined for any $v \in F_N$ by $i_w(v) = w^{-1}vw$. Let φ be an A_N -positive primitive automorphism, and let Σ_φ be its attracting subshift.

Definition 3.1. A φ -singularity is a set Ω of (pairwise distinct) points of Σ_φ satisfying the following conditions:

- Ω contains at least two elements,
- there exists an automorphism $\psi = i_w \circ \varphi^k$ for some conjugacy i_w and some integer $k \geq 1$ such that all the points of Ω are fixed points of $\partial^2 \psi$,
- for any $h \in \mathbb{N}^*$, if $(U, V) \in \Sigma_\varphi$ is a fixed point of $\partial^2 \psi^h$, then $(U, V) \in \Omega$,
- there exist two points (U, V) and (U', V') of Ω such that we have either $V_0 \neq V'_0$ or $U_0 \neq U'_0$ or both.

We say that the automorphism ψ fixes the singularity Ω .

Let δ be a self-induced CIET and φ its δ -automorphism. Observe that for any δ -singularity x , the set $\Sigma_\delta(x)$ is a φ -singularity: we can easily deduce that there must exist an automorphism $i_w \circ \varphi^k$ such that $\partial^2(i_w \circ \varphi^k)$ fixes all the points of $\Sigma_\delta(x) = \{(U, V), (U', V')\}$ by noticing that since $U = U'$ (resp. $V = V'$) if x is a forward (resp. backward) δ -singularity, we also have $\partial^2 \varphi(U, V) = (X, Y)$ and $\partial^2 \varphi(U', V') = (X', Y')$ with $X = X'$ (resp. $Y = Y'$). In fact, this is also true for $k = 1$.

Proposition 3.2. *For any singularity Ω , there exists $w \in F_N$ such that $i_w \circ \varphi$ fixes Ω .*

Proof. Define $\delta = (\pi, \lambda)$. Let n be the smallest integer such that $R^n(\delta) = \delta' = (\pi', \lambda')$ with $\pi' = \pi$ and $\lambda' = \eta\lambda$ for some positive real number η . Recall (proposition 2.6) that for any point $x \in [0, |\lambda'|]$, we have $\Sigma_\delta(x) = \partial^2 \varphi(\Sigma_{\delta'}(x))$. We deduce from $\pi = \pi'$ together with proposition 2.7 that if x is the i th (from the left) forward (resp. backward) δ' -singularity, then there exists a word $u = u_0 u_1 \dots u_p$ with letters in A_N (resp. A_N^{-1}) such that $\delta_{u_p} \circ \dots \circ \delta_{u_0}(x)$ is the i th (from the left) forward (resp. backward) δ -singularity. This last property tells us any φ -singularity Ω is globally invariant by $i_w \circ \varphi$ for some w : we have $\partial^2(i_w \circ \varphi)(\Omega) = \Omega$. We conclude each point of Ω is in fact fixed by using $\pi = \pi'$ again. \square

We now have the tools to adress the problem raised by the discussion below theorem 2.8.

Proposition 3.3. *Let $\delta = (\pi, \lambda)$ be a self-induced CIET and let $\delta' = (\pi', \lambda)$ be such that, for any $a \in A_N$, we have $\pi'_0(a) = N - 1 - \pi_0(a)$ and $\pi'_1(a) = N - 1 - \pi_1(a)$. Define φ as the δ -automorphism, $\Sigma_\delta(0) = \{Z\}$ and $\Sigma_\delta(|\lambda|) = \Sigma_{\delta'}(0) = \{Z'\}$. The CIET δ' is self-induced if and only if the automorphism $i_u \circ \varphi$ such that $\partial^2(i_u \circ \varphi)(Z') = Z'$ is A_N -positive.*

Proof. Note that the existence of a word $u \in F_N$ such that $\partial^2(i_u \circ \varphi)(Z') = Z'$ is a consequence of proposition 3.2. If δ' is self-induced, we deduce $i_u \circ \varphi$ is A_N -positive from the definitions of the δ -automorphism and δ' -automorphism along with proposition 3.5 (see further).

Assume $\psi = i_u \circ \varphi$ is A_N -positive and consider the CIET δ' on the interval $[0, |\lambda|]$. For any $a \in A_N$, define $I_a \subset [0, |\lambda|]$ (resp. $I_{a^{-1}} \subset [0, |\lambda|]$) as the set of points x such that there exists $(U, V) \in \Sigma_{\delta'}$ with $V_0 = a$ (resp. $U_0 = a^{-1}$) such that $\partial^2\psi(U, V) \in \Sigma_{\delta'}(x)$. Define the system of partial isometries $\delta'' = (\delta''_a)_{a \in A_N}$ where for any $a \in A_N$, the map $\delta''_a : I_a \rightarrow I_{a^{-1}}$ is a translation. We want to show that $\delta'' = R^n(\delta')$ for some positive integer n . First, we prove δ'' is effectively a CIET. For any $a \in A_N$, the set I_a (resp. $I_{a^{-1}}$) is a closed interval; if it were not, the domain of δ''_a (resp. $\delta''_{a^{-1}}$) would not be a closed interval either. Define $Z' = (X', Y')$. Since $\partial^2\psi(Z') = Z'$, both $I_{Y'_0}$ and $I_{X'_0}$ contain 0. We deduce from proposition 3.2 that for any $a, b \in A_N$, $I_a \cap I_b$ (resp. $I_{a^{-1}} \cap I_{b^{-1}}$) contains (exactly) one point if and only if $|\pi_0(a) - \pi_0(b)| = 1$ (resp. $|\pi_1(a) - \pi_1(b)| = 1$) and is empty otherwise. Hence, δ'' is a CIET and we have $\delta'' = (\pi'', \lambda'')$ with $\pi'' = \pi'$. Moreover, we have $\lambda'' = \eta\lambda'$ for some positive real number η ; if this were not the case, we would easily deduce $\Sigma_{\delta''}(0) \neq \{Z'\}$ from minimality. Also, as a consequence of [CS01, proposition 6.2], δ'' is a first return system (with respect to remark 2.5). We conclude by using [Vee82, proposition 8.9] and [Rau79, theorem 23] which state a first return system on $[0, |\lambda''|]$ has (exactly) $N - 1$ singularities if and only if it can be obtained by Rauzy inductions. \square

3.1 The necessary conditions

Let $\delta = (\pi, \lambda)$ be a self-induced CIET and let φ be its δ -automorphism. We explicit a list of necessary conditions that are satisfied by both φ and Σ_φ . They will be the starting point of our algorithm. Some of these conditions are deliberately overly detailed to prepare for the algorithm.

- (C 1) Following from proposition 2.1, there is exactly $2N - 2$ distincts φ -singularities $\{\Omega_0, \dots, \Omega_{2N-3}\}$, each containing exactly 2 points.
- (C 2) From corollary 2.2, we can order the φ -singularities so that for any $0 \leq i \leq N - 2$,

$$\Omega_i = \{(U_{(i)}, V_{(i)}), (U'_{(i)}, V'_{(i)})\} \text{ with } U_{(i)} = U'_{(i)}$$

and for any $N - 1 \leq j \leq 2N - 3$,

$$\Omega_j = \{(U_{(j)}, V_{(j)}), (U'_{(j)}, V'_{(j)})\} \text{ with } V_{(j)} = V'_{(j)}.$$

(C 3) Define the **forward graph** G_+ (resp. **backward graph** G_-) as the graph whose nodes are the elements of A_N and there is an (unoriented) edge from a to b if there exists $0 \leq i \leq N-2$ (resp. $N-1 \leq j \leq 2N-3$) such that a and b (resp. a^{-1} and b^{-1}) are the first letters of $V_{(i)}$ and $V'_{(i)}$ (resp. $U_{(j)}$ and $U'_{(j)}$); such an edge is labeled by a_0 if a_0^{-1} (resp. a_0) is the first letter of $U_{(i)}$ (resp. $V_{(j)}$). Define the **distance** between two nodes (resp. a node and an edge) as the number of edges contained in the path joining one to the other (note: conventionnally, the path joining a node a to an edge e does not contain e). We get the following conditions from theorem 2.8.

(C 3.1) Both graphs are connected and each one contains two nodes of degree (the number of adjacent edges) 1 while all the others have degree 2.

(C 3.2) Observe that $\pi_0^{-1}(0) = \alpha$ (resp. $\pi_1^{-1}(0) = \beta$) is a node of G_+ (resp. G_-) with degree 1. Moreover, for any node a of G_+ (resp. G_-), $\pi_0(a)$ (resp. $\pi_1(a)$) is given by the distance between a and α (resp. a and β).

(C 3.3) Let e_a and e_b be two edges of G_+ (resp. G_-) labeled a and b respectively and suppose $a \neq b$. Then $\pi_1(a) < \pi_1(b)$ (resp. $\pi_0(a) < \pi_0(b)$) if and only if e_a is closer to α (resp. β) than e_b . It may happen that all the edges of G_+ have a common label β_0 and all the edges of G_- have a common label β_1 . In that case, we prove the following proposition.

Proposition 3.4. *We have $\beta_0 = \beta_1$ and $\{\pi_0(\beta_0), \pi_1(\beta_0)\} = \{0, N-1\}$.*

Proof. Since all the forward δ -singularities are contained in the domain $[y_{\beta_0}, y'_{\beta_0}]$ of $\delta_{\beta_0^{-1}}$, there exist $a, b \in A_N$ with $a \neq b$ such that y_{β_0} (resp. y'_{β_0}) is in the domain of δ_a (resp. δ_b). If $\pi_1(\beta_0)$ is neither 0 nor $N-1$, then both y_{β_0} and y'_{β_0} are backward δ -singularities and we have a contradiction. Obviously, the same reasoning tells us $\pi_0(\beta_1)$ must also be either 0 or $N-1$. We deduce $\lambda_{\beta_0} > \sum_{a \neq \beta_0} \lambda_a$ and $\lambda_{\beta_1} > \sum_{a \neq \beta_1} \lambda_a$ and conclude $\beta_0 = \beta_1$.

Finally, we observe the equality $\pi_0(\beta_0) = \pi_1(\beta_0)$ contradicts the Keane condition. \square

(C 4) Define $\alpha_0 = \pi_0^{-1}(N-1)$ and $\alpha_1 = \pi_1^{-1}(N-1)$. There is exactly one point (U, V) of one φ -singularity Ω such that $U_0^{-1} = \alpha_1$ and $V_0 = \alpha_0$. The φ -singularity Ω contains a point (U', V) with $U' \neq U$ if and only if δ has type 0.

(C 5) If $\Sigma_\delta(0) = \{Z\}$, then $\partial^2 \varphi(Z) = Z$.

Observe that different A_N -positive primitive automorphisms may have the same attracting subshift. In particular, (see [BFH97]) if φ and ψ are two A_N -positive primitive automorphisms with $i_u \circ \psi^k = \varphi^h$ for some $u \in F_N$ and $k, h \geq 1$, then $\Sigma_\varphi = \Sigma_\psi$. It is unclear in the general case if other automorphisms sharing the attracting subshift may be found. The following proposition deals with the issue and also ensures that the δ -automorphisms possess a minimality property; none of his roots fixes all the φ -singularities.

Proposition 3.5. *Let ψ be an A_n -positive automorphism such that $\Sigma_\psi = \Sigma_\varphi$ and let k be a positive integer such that any φ -singularity Ω is fixed by $i_w \circ \psi^k$ for some $w \in F_N$. Then there exists $u \in F_N$ such that $\partial^2(i_u \circ \psi^k)(Z) = Z$ and we have $i_u \circ \psi^k = \varphi^h$ for some positive integer h .*

Proof. First, note that the set of ψ -singularities is the set of φ -singularities: this is obvious since $\Sigma_\psi = \Sigma_\varphi$. Define $Z = (X, Y)$ and observe there exists a backward φ -singularity containing the point $(U, V) = Y_0^{-1}Z$ (where Y_0 is the first letter of Y). We know there exists $v \in F_N$ such that $\partial^2(i_v \circ \psi^k)(U, V) = (U, V)$ and we deduce $\partial^2(i_u \circ \psi^k)(Z) = Z$ for $u = \psi^k(Y_0)vY_0^{-1}$.

In order to prove that $i_u \circ \psi^k = \varphi^h$ for some $h > 0$, we are going to use results on pseudo-Anosov homeomorphisms on translation surfaces. We refer to [Vee82] for the original construction of pseudo-Anosov homeomorphisms using interval exchange transformations and to [BL10] for a nice summary of the necessary definitions and results. General results on homeomorphisms on surfaces may also be found in [FLP79].

A homeomorphism f on a surface \mathcal{S} (we assume it has a boundary component) is pseudo-Anosov if there exist two transversely measured foliations (\mathcal{F}_s, μ_s) and (\mathcal{F}_u, μ_u) , respectively called the stable and unstable foliations, such that $f(\mathcal{F}_s) = \eta^{-1}\mathcal{F}_s$ and $f(\mathcal{F}_u) = \eta\mathcal{F}_u$ for some real number $\eta > 1$. The action of the map f on the fundamental group of S can be seen as a free group automorphism whose attracting subshift is a combinatorial interpretation of the stable foliation.

We associate (as in [Vee82] or [BL10]) a translation surface S_δ to our self-induced CIET δ . In [Vee82], Veech gives an interpretation on S_δ of the Rauzy induction; the transformation is referred to as the Rauzy-Veech induction. The following result is attributed to Veech in [BL10]. It should be understood up to composition by a conjugacy.

Theorem 3.6 ([Vee82]). *All pseudo-Anosov homeomorphisms of S_δ that fix a separatrix are obtained by Rauzy-Veech inductions.*

A separatrix is a half leaf of the stable foliation that is attached to a singularity of S_δ . A point $x \in S_\delta$ belonging to a leaf L is a singularity if the combinatorial interpretation of L with marked point x is a point (U', V') of a φ -singularity; the left end point and the right end point of the interval are also considered singularities. Hence, a separatrix is interpreted combinatorially as a point $X' \in \partial F_N$ such that there exists $Y' \in \partial F_N$ such that (X', Y') (resp. (Y', X')) belongs to a φ -singularity or is the orbit of the left end point or right end point of the interval; such a point is called a combinatorial separatrix.

It is possible that the homeomorphism $\partial\psi^k$ (induced on ∂F_N by ψ^k) does not fix a combinatorial separatrix. In that case, we prove that we can work with an A_N -positive automorphism $\psi' = i_v \circ \psi^k$ that fixes a φ -singularity ($\partial\psi'$ will then fix combinatorial separatrix).

Recall we assumed any φ -singularity is fixed by $i_w \circ \psi^k$ for some w . Suppose $\partial\psi^k$ does not fix a combinatorial separatrix. Then ψ^k does not fix a φ -singularity. We also deduce that there must exist a letter $a_0 \in A_N$ such that a_0 is the first letter of $\psi^k(a_0)$ (otherwise there would be a φ -singularity that is fixed by some power of ψ^k but not by ψ^k). Moreover, if $i_w \circ \psi^k$ fixes the φ -singularity $\Omega = \{(X, Y), (X, Y')\}$, then the first letter of w is also the first letter of both $\psi^k(Y_0)$ and $\psi^k(Y'_0)$. Applying this to all φ -singularities, we deduce a_0 is the first letter of $\psi^k(a)$ for any $a \in A_N$. Similarly, one can prove there exist a letter b_0 which is the last letter of $\psi^k(a)$ for any $a \in A_N$. We can then obtain an A_N -positive automorphism $\psi' = i_v \circ \psi^k$ that fix a φ -singularity.

We simply assume ψ^k fixes a singularity. Both φ and ψ^k are seen as the action of pseudo-Anosov homeomorphisms fixing separatrix on the fundamental group of S_δ , and we deduce from theorem 3.6 that there exist two positive integers i and j such that $(i_u \circ \psi^k)^i = \varphi^j$. We conclude with the definition of the δ -automorphism and proposition 3.2. \square

We conjecture that condition (C 1) is a (necessary and) sufficient condition for the attracting subshift of an A_N -positive primitive automorphism to be the subshift of a CIET.

Conjecture 3.7. *If ψ is an A_N -positive primitive automorphism satisfying condition (C 1), then there exists a self-induced CIET δ such that $\Sigma_\delta = \Sigma_\psi$. In particular, ψ also satisfies (C 2) and (C 3) and if φ is the δ -automorphism, then $i_u \circ \psi^k = \varphi^h$ for some $h, k \geq 1$ and $u \in F_N$.*

3.2 The algorithm

The algorithm is based on the ability to identify the singularities of an A_N -positive primitive automorphism. The reader is referred to [Jul10] for a complete approach of the problem; the relevant results of [Jul10] are summarized in appendix A.

The algorithm does not pretend to be optimal. It may surely be improved in many ways, as there are many more necessary conditions than the ones we have listed. In fact, the aim here is to present a minimal list of such conditions. Another good point is that the approach we use is fully combinatorial.

Let ψ be an A_N -positive primitive automorphism. The following algorithm will determine in a finite time if its attracting subshift Σ_ψ is equal to Σ_δ for some CIET δ . An example is detailed in appendix B.

- (1) List the ψ -singularities using the algorithm of [Jul10]. Stop if conditions (C 1) and (C 2) are not both satisfied.
- (2) Define the forward graph G_+ and the backward graph G_- as in condition (C 3), and stop if condition (C 3.1) is not satisfied. We now define a

pair $\pi = (\pi_0, \pi_1)$ with $\pi_0, \pi_1 : A_N \rightarrow \{0, \dots, N-1\}$ that will agree with condition (C 3.2). Choose a node α with degree 1 in G_+ . For any $a \in A_N$, define $\pi_0(a)$ as the distance between a and α . If G_+ contains two edges with distinct labels, apply (2.1). If all the edges of G_+ have the same label and G_- contains two edges with distinct labels, apply (2.2). Apply (2.3) otherwise.

- (2.1) Choose two edges e_a and e_b of G_+ labeled a and b with $a \neq b$ and such that e_a is closer to α than e_b . Choose the node β of degree 1 of G_- such that a is closer to β than b . For any $a_0 \in A_N$, define $\pi_1(a_0)$ as the distance between the nodes a_0 and β .
- (2.2) Choose two edges e_a and e_b of G_- labeled a and b with $a \neq b$ and such that a is closer to α than b in G_+ . Choose the node β of degree 1 of G_- such that e_a is closer to β than e_b . For any $a_0 \in A_N$, define $\pi_1(a_0)$ as the distance between the nodes a_0 and β .
- (2.3) Suppose all the edges of G_+ have the same label β_0 and all the edges of G_- have the same label β_1 . Stop if $\beta_0 \neq \beta_1$. Also stop if α does not have degree 1 in G_- . For any $a_0 \in A_N$ define $\pi_1(a_0) = N-1-k$ where k is the distance between a_0 and α (in G_-).

In any case, stop if conditions (C 3.2), (C 3.3) and (C 4) are not all satisfied. We also check if π is reducible and stop if it is.

- (3) We now deal with condition (C 5). Choose the smallest positive integer k such that any ψ -singularity is fixed by $i_w \circ \psi^k$ for some $w \in F_N$. Let $\Omega = \{(U', V), (U, V)\}$ be the backward ψ -singularity such that $U_0 = (\pi_0^{-1}(0))^{-1}$ and $\pi_1(U_0'^{-1}) = \pi_1(U_0^{-1}) - 1$. Define $Z = U_0^{-1}(U, V)$ and $u \in F_N$ such that $\partial^2(i_u \circ \psi^k)(Z) = Z$; also define $\varphi' = i_u \circ \psi^k$.

Following from the discussion below theorem 2.8, we may also want to consider the pair π' defined for any $a \in A_N$ by $\pi'_0(a) = N-1-\pi_0(a)$ and $\pi'_1(a) = N-1-\pi_1(a)$. Let $\Omega' = \{(X', Y), (X, Y)\}$ be the backward ψ -singularity such that $X_0 = (\pi_0'^{-1}(0))^{-1}$ and $\pi'_1(X_0'^{-1}) = \pi'_1(X_0^{-1}) - 1$. Define $Z' = X_0^{-1}(X, Y)$ and $v \in F_N$ such that $\partial^2(i_v \circ \psi^k)(Z') = Z'$; also define $\varphi'' = i_v \circ \psi^k$.

According to proposition 3.5, we can stop if neither φ' nor φ'' is A_N -positive. If φ' (resp. φ'') is A_N -positive, we define $\varphi = \varphi'$ (resp. $\varphi = \varphi''$). Thanks to proposition 3.3, we may define φ to be either one of them if they are both A_N -positive.

- (4) Set up a counter $i \leftarrow 0$. Define $\{\Omega_j^{(i)}, 0 \leq j \leq 2N-3\}$ as the set of ψ -singularities, $(\pi_0^{(i)}, \pi_1^{(i)}) = (\pi_0, \pi_1)$ and $G_+^{(i)} = G_+$.
- (5) Define $\alpha_0^{(i)} = (\pi_0^{(i)})^{-1}(N-1)$ and $\alpha_1^{(i)} = (\pi_1^{(i)})^{-1}(N-1)$ and let e be the (only) edge of $G_+^{(i)}$ adjacent to $\alpha_0^{(i)}$.

- (5.1) If e is not labeled $\alpha_1^{(i)}$ define the automorphism σ_i such that $\sigma_i(\alpha_1^{(i)}) = \alpha_1^{(i)}\alpha_0^{(i)}$ and $\sigma_i(a) = a$ for any $a \neq \alpha_1^{(i)}$.
- (5.2) If e is labeled $\alpha_1^{(i)}$, define the automorphism σ_i such that $\sigma_i(\alpha_0^{(i)}) = \alpha_1^{(i)}\alpha_0^{(i)}$ and $\sigma_i(a) = a$ for any $a \neq \alpha_0^{(i)}$.
- (6) • If the automorphism $\sigma_i^{-1} \circ \dots \circ \sigma_0^{-1} \circ \varphi$ is the identity, the algorithm is a success. Define the incidence matrix M of ψ^k as the matrix $N \times N$ where $M_{a,b}$ is defined for any $a, b \in A_N$ as the number of occurrences of the letter a in $\psi^k(b)$. The matrix M is primitive (one of its power only has positive entries) since ψ^k is A_N -primitive. Perron-Frobenius theorem tells us the eigenvalue with greatest modulus is simple, has modulus (strictly) greater than the modulus of the other eigenvalues, is a positive real number and has a positive eigenvector λ . Define $\delta = (\pi, \lambda)$ where π is as above. We have $\Sigma_\psi = \Sigma_\delta$. The smallest period of the sequence $(\sigma_j)_{0 \leq j \leq i}$ will give the δ -automorphism.
- If the automorphism $\sigma_i^{-1} \circ \dots \circ \sigma_0^{-1} \circ \varphi$ is not the identity but is A_N -positive, we continue on to step (7). Observe any letter of A_N appears in $\sigma_{i-1}^{-1} \circ \dots \circ \sigma_0^{-1} \circ \varphi(a)$ for at least one letter $a \in A_N$ (otherwise $\sigma_{i-1}^{-1} \circ \dots \circ \sigma_0^{-1} \circ \varphi$ would not be an automorphism) and deduce that the automorphism $\sigma_i^{-1} \circ \dots \circ \sigma_0^{-1} \circ \varphi$ is (strictly) shorter (with respect to the lengths of the images of elements of A_N) than $\sigma_{i-1}^{-1} \circ \dots \circ \sigma_0^{-1} \circ \varphi$, ensuring the algorithm will effectively end after a finite, easily bounded, number of steps.
- According to proposition 3.5, we can stop if the automorphism $\sigma_i^{-1} \circ \dots \circ \sigma_0^{-1} \circ \varphi$ is not A_N -positive.
- (7) We define a new set of singularities. We assume $\Omega_k^{(i)}$ contains the (only) point (U, V) with $U_0^{-1} = \alpha_1^{(i)}$ and $V_0 = \alpha_0^{(i)}$. For any $1 \leq j \leq 2N - 3$, $j \neq k$, we simply define $\Omega_j^{(i+1)} = \partial^2 \sigma_i^{-1}(\Omega_j^{(i)})$. Observe that if σ_i is defined as in step (5.1) (resp. (5.2)), then $\Omega_k^{(i)}$ contains a point (U', V) with $U' \neq U$ (resp. (U, V') with $V \neq V'$) and apply step (7.1) (resp. (7.2)).
- (7.1) Define $\Omega_k^{(i+1)} = \partial^2 \sigma_i^{-1}(S(\Omega_k^{(i)}))$. Define $(\pi_0^{(i+1)}, \pi_1^{(i+1)})$ as the result of a type 0 induction on $(\pi_0^{(i)}, \pi_1^{(i)})$. Namely, $\pi_0^{(i+1)} = \pi_0^{(i)}$ and
- $\forall a \in A_N; \pi_1^{(i)}(a) \leq \pi_1^{(i)}(\alpha_0^{(i)}), \pi_1^{(i+1)}(a) = \pi_1^{(i)}(a),$
 - $\pi_1^{(i+1)}(\alpha_1^{(i)}) = \pi_1^{(i)}(\alpha_0^{(i)}) + 1,$
 - $\forall a \in A_N; \pi_1^{(i)}(\alpha_0^{(i)}) < \pi_1^{(i)}(a) < \pi_1^{(i)}(\alpha_1^{(i)}), \pi_1^{(i+1)}(a) = \pi_1^{(i)}(a) + 1.$
- (7.2) Define $\Omega_k^{(i+1)} = \partial^2 \sigma_i^{-1}(S^{-1}(\Omega_k^{(i)}))$. Define $(\pi_0^{(i+1)}, \pi_1^{(i+1)})$ as the result of a type 1 induction on $(\pi_0^{(i)}, \pi_1^{(i)})$. Namely, $\pi_1^{(i+1)} = \pi_1^{(i)}$ and
- $\forall a \in A_N; \pi_0^{(i)}(a) \leq \pi_0^{(i)}(\alpha_1^{(i)}), \pi_0^{(i+1)}(a) = \pi_0^{(i)}(a),$
 - $\pi_0^{(i+1)}(\alpha_0^{(i)}) = \pi_0^{(i)}(\alpha_1^{(i)}) + 1,$

- $\forall a \in A_N; \pi_0^{(i)}(\alpha_1^{(i)}) < \pi_0^{(i)}(a) < \pi_0^{(i)}(\alpha_0^{(i)}), \pi_0^{(i+1)}(a) = \pi_0^{(i)}(a) + 1.$

Define the new graphs $G_+^{(i+1)}$ and $G_-^{(i+1)}$ and note that conditions (C 1), (C 2), (C 3.1) and (C 3.2) are automatically satisfied. Stop if conditions (C 3.3) and (C 4) are not both satisfied. Set $i \leftarrow i + 1$ and go back to step (5).

A Identifying the singularities of an A_N -positive primitive automorphism

The article [Jul10] gives an algorithm to find the singularities of an A_N -positive primitive automorphism. The main tool of the algorithm is a careful study of the prefix-suffix representation of an attracting subshift. The reader is referred to [CS01] for results on the prefix-suffix representation. We briefly recall here the slightly adapted version of [Jul10].

Let φ be an A_N -positive primitive automorphism and Σ_φ its attracting subshift. The **prefix-suffix automaton** of φ is defined as follows:

- A_N is its set of vertices,
- $P = \{(p, a, s) \in A_N^* \times A_N \times A_N^*; \exists b \in A_N; \varphi(b) = pas\}$ (where A_N^* is the set of words of F_N with letters in A_N and $\epsilon \in A_N^*$ is the set of labels,
- there is an edge labeled (p, a, s) from a to b if and only if $\varphi(b) = pas$.

The set of sequences of labels of infinite walks in this automaton is denoted D ; it is the set of admissible developments.

Proposition A.1. *If $(p_i, a_i, s_i)_{i \geq 0}$ is in D , then for all $n \in \mathbb{N}$, $\varphi(a_{n+1}) = p_n a_n s_n$.*

In [CS01], the authors define a map $\rho_\varphi : \Sigma_\varphi \rightarrow D$ which gives a representation of the attracting subshift's structure. If $W = (U, V)$ is a point of Σ_φ , then $\rho_\varphi(W) = (p_i, a_i, s_i)_{i \geq 0}$ is called the **prefix-suffix development** of W , and it is such that

- if $(s_i)_{i \in \mathbb{N}}$ is not eventually constant equal to ϵ , then

$$V = \lim_{n \rightarrow +\infty} a_0 s_0 \varphi(s_1) \dots \varphi^n(s_n),$$

- if $(p_i)_{i \in \mathbb{N}}$ is not eventually constant equal to ϵ , then

$$U = \lim_{n \rightarrow +\infty} p_0^{-1} \varphi(p_1^{-1}) \dots \varphi^n(p_n^{-1}).$$

Those developments whose prefix or suffix sequence end up being constant equal to ϵ are identified in [CS01].

Theorem A.2 ([CS01]). *The map ρ_φ is continuous and onto. Any development $d \in D$ has at most N pre-images.*

We now recall the relevant results of [Jul10]. We still consider φ is an A_N -positive primitive automorphism and Σ_φ is its attracting subshift. Observe that for any positive integer k , the automorphism φ^k is also A_N -positive and primitive and its attracting subshift Σ_{φ^k} verifies $\Sigma_{\varphi^k} = \Sigma_\varphi$. For any $k \geq 1$, let $\gamma_{\varphi_-^k}$ and $\gamma_{\varphi_+^k}$ be the F_N to F_N maps defined, for every $u = u_0 u_1 \dots u_p$ in F_N , by

- $\gamma_{\varphi_-^k}(u) = \varphi^k(u_p) u_0 u_1 \dots u_{p-1}$,
- $\gamma_{\varphi_+^k}(u) = u_1 \dots u_{p-1} u_p \varphi^k(u_0)$.

The search for φ -singularities is based on the following result.

Theorem A.3 ([Jul10]). *Let W and W' be two distinct points of Σ_φ with $\rho_{\varphi^k}(W) = (p, a, s)^*$ and $\rho_{\varphi^k}(W') = (q, b, r)^*$ (where ρ_{φ^k} is the prefix-suffix development map of φ^k and the symbol $*$ indicates the triplet is repeated indefinitely).*

- *If for any $i, j \in \mathbb{N}$, we have $\gamma_{\varphi_-^k}^i(p) \neq \gamma_{\varphi_-^k}^j(q)$ (resp. $\gamma_{\varphi_+^k}^i(s) \neq \gamma_{\varphi_+^k}^j(r)$), then for any $i, j \in \mathbb{N}$, the points $S^{-i}(W)$ and $S^{-j}(W')$ (resp. $S^{i+1}(W)$ and $S^{j+1}(W')$) do not belong to a common singularity.*
- *If i and j are the smallest integers such that $\gamma_{\varphi_-^k}^i(p) = \gamma_{\varphi_-^k}^j(q)$ (resp. $\gamma_{\varphi_+^k}^i(s) = \gamma_{\varphi_+^k}^j(r)$), then $S^{-i}(W)$ and $S^{-j}(W')$ (resp. $S^{i+1}(W)$ and $S^{j+1}(W')$) belong to the same singularity Ω . Moreover, the singularity Ω is fixed by $(i_w \circ \varphi)^h$ for some integer $h \geq 1$ and $w = \gamma_{\varphi_-^k}^i(p)$ (resp. $w^{-1} = \gamma_{\varphi_+^k}^i(s)$).*

Conversely, if $W_{(0)}$ and $W_{(1)}$ are two points of Σ_φ belonging to a common singularity, then there exist an integer $k \leq 4N - 4$ and two points W and W' of Σ_φ such that

- $\rho_{\varphi^k}(W) = (p, a, s)^*$, $\rho_{\varphi^k}(W') = (q, b, r)^*$,
- $\gamma_{\varphi_-^k}^i(p) = \gamma_{\varphi_-^k}^j(q)$ (resp. $\gamma_{\varphi_+^k}^i(s) = \gamma_{\varphi_+^k}^j(r)$) for some integers $i, j \geq 0$,
- $W_{(0)} = S^{-i}(W)$ and $W_{(1)} = S^{-j}(W')$ (resp. $W_{(0)} = S^{i+1}(W)$ and $W_{(1)} = S^{j+1}(W')$).

Keeping the notation of the theorem above, note that it is possible for $S^{-i}(W)$ and $S^{-j}(W')$ (resp. $S^{i+1}(W)$ and $S^{j+1}(W')$) to belong to the same singularity even if i and j are not the smallest integers for which $\gamma_{\varphi_-^k}^i(p) = \gamma_{\varphi_-^k}^j(q)$ (resp. $\gamma_{\varphi_+^k}^i(s) = \gamma_{\varphi_+^k}^j(r)$). This is typical of singularities containing two distinct points (U, V) and (U', V') with both $U_0 = U'_0$ and $V_0 = V'_0$ (an example is given in [Jul10]).

Recall that different points may have the same prefix-suffix development. As theorem A.3 only take the prefix-suffix development into consideration, if using theorem A.3 tells us $S^m(W)$ and $S^n(W')$ belong to Ω , then we also know

that for any point $W_{(0)}$ (resp. $W'_{(0)}$) such that $\rho_{\varphi^k}(W) = \rho_{\varphi^k}(W_{(0)})$ (resp. $\rho_{\varphi^k}(W') = \rho_{\varphi^k}(W'_{(0)})$), the point $S^m(W_{(0)})$ (resp. $S^n(W'_{(0)})$) belongs to Ω .

The algorithm to find φ -singularities simply consists, for each $1 \leq k \leq 4N - 4$, in running pairs $((p, a, s)*, (q, b, r)*)$ of constant (with respect to ρ_{φ^k}) prefix-suffix developments through theorem A.3. It is obvious from proposition A.1 that there is a finite number of such developments. Moreover, it is explained in [Jul10] how properties of φ^{-k} can be used to bound the minimal integers i, j such that $\gamma_{\varphi_-^k}^i(p) = \gamma_{\varphi_-^k}^j(q)$ (resp. $\gamma_{\varphi_+^k}^i(s) = \gamma_{\varphi_+^k}^j(r)$), ensuring the algorithm will end after a finite number of steps.

Note that it may not be necessary to sweep through all constant prefix-suffix developments pairs, as the overall number of points contained in singularities is bounded (see [GJLL98], [Jul10]). This bound is reached for CIETs.

Remark A.4. One may be interested in finding the φ -singularities when φ is the δ -automorphism of some CIET δ . From proposition 3.2, we only need to study developments that are constant for ρ_{φ} .

B An example

Let ψ be the $\{a, b, c, d\}$ -positive primitive automorphism defined by

$$\begin{array}{rcl} \psi & : & a \mapsto bdacda \\ & & b \mapsto bdbda \\ & & c \mapsto ccda \\ & & d \mapsto cda \end{array}$$

There are a lot of obvious pairs yielding ψ -singularities. We get:

- $\Omega_0 = \{W_{(0)}, W'_{(0)}\}$ with $\rho_{\psi}(W_{(0)}) = (\epsilon, b, dbda)*$ and $\rho_{\psi}(W'_{(0)}) = (\epsilon, c, cda)*$,
- $\Omega_1 = \{W_{(1)}, W'_{(1)}\}$ with $\rho_{\psi}(W_{(1)}) = (bd, a, cda)*$ and $\rho_{\psi}(W'_{(1)}) = (bd, b, da)*$,
- $\Omega_2 = \{W_{(2)}, W'_{(2)}\}$ with $\rho_{\psi}(W_{(2)}) = (c, c, da)*$ and $\rho_{\psi}(W'_{(2)}) = (c, d, a)*$,
- $\Omega_3 = \{W_{(3)}, W'_{(3)}\}$ with $\rho_{\psi}(S^{-1}(W_{(3)})) = (bd, a, cda)*$ and $\rho_{\psi}(S^{-1}(W'_{(3)})) = (\epsilon, c, cda)*$,
- $\Omega_4 = \{W_{(4)}, W'_{(4)}\}$ with $\rho_{\psi}(S^{-1}(W_{(4)})) = (bd, b, da)*$ and $\rho_{\psi}(S^{-1}(W'_{(4)})) = (c, c, da)*$.

Also, observe $\gamma_{\psi_+}(a) = \gamma_{\psi_+}(bdba)$, and deduce there is a singularity $\Omega_5 = \{W_{(5)}, W'_{(5)}\}$ with $\rho_{\psi}(S^{-2}(W_{(5)})) = (c, d, a)*$ and $\rho_{\psi}(S^{-2}(W'_{(5)})) = (\epsilon, b, dbda)*$.

Remark B.1. Define $U = \lim_{n \rightarrow +\infty} \psi^n(a^{-1})$. The point U is the first coordinate of the points $W_{(0)}$, $W'_{(0)}$, $S^{-1}(W'_{(3)})$ and $S^{-2}(W'_{(5)})$. All the other coordinates are explicitly given by the prefix-suffix developments.

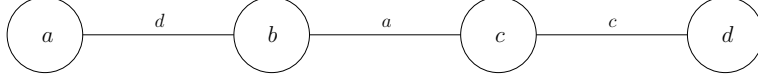


Figure 4: First forward graph.

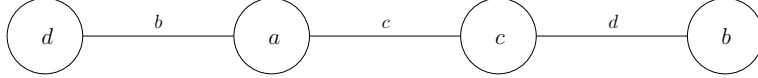


Figure 5: First backward graph.

The forward and backward graphs are given on figures 4 and 5. Using step (2.1), we can define the pair $\pi = (\pi_0, \pi_1)$:

$$\begin{array}{lll} \pi_0 : & a & \mapsto 0 \\ & b & \mapsto 1 \\ & c & \mapsto 2 \\ & d & \mapsto 3 \end{array} \quad \begin{array}{lll} \pi_1 : & a & \mapsto 1 \\ & b & \mapsto 3 \\ & c & \mapsto 2 \\ & d & \mapsto 0 \end{array}$$

Let (U, V) be the point of Ω_5 such that $U_0 = a^{-1}$; define $Z = a(U, V)$. The singularity Ω_5 is fixed by $i_{(bdacda)^{-1}} \circ \psi$, and we deduce Z is fixed by $\partial^2(i_{a^{-1}} \circ i_{(bdacda)^{-1}} \circ \psi \circ i_a)$. Define $\varphi = i_{a^{-1}} \circ i_{(bdacda)^{-1}} \circ \psi \circ i_a = i_{\psi(a)(abdacda)^{-1}} \circ \psi = i_{a^{-1}} \circ \psi$. We obtain the $\{a, b, c, d\}$ -positive automorphism

$$\begin{array}{lll} \varphi : & a & \mapsto abdacd \\ & b & \mapsto abdbd \\ & c & \mapsto accd \\ & d & \mapsto acd \end{array}$$

Note that, in this case, working with $\pi' = (\pi'_0, \pi'_1)$ defined by $\pi'_0 = 3 - \pi_0(a_0)$ and $\pi'_1 = 3 - \pi_1(a_0)$ for any $a_0 \in \{a, b, c, d\}$ would not have provided us with an $\{a, b, c, d\}$ -positive automorphism.

From step (5), we define $\sigma_0(b) = bd$ and $\sigma_0(a_0) = a_0$ if $a_0 \neq b$. We obtain

$$\begin{array}{lll} \sigma_0^{-1} \circ \varphi : & a & \mapsto abacd \\ & b & \mapsto abb \\ & c & \mapsto accd \\ & d & \mapsto acd \end{array}$$

which is still A_N -positive, and continue on to step (7). Obviously, we only need the first few letters of the coordinates of the points contained in the singularities to move on. Following from step (7) (in this case (7.1)), we obtain the new graphs of figures 6 and 7.

This allows us (step (5)) to define $\sigma_1(d) = cd$ and $\sigma_1(a_0) = a_0$ if $a_0 \neq d$, and we obtain

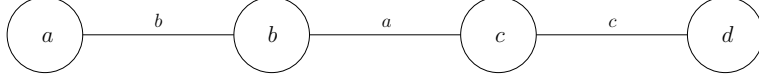


Figure 6: Second forward graph.

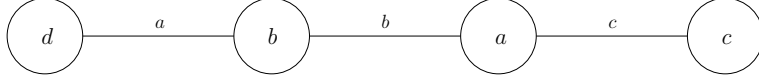


Figure 7: Second backward graph.

$$\begin{array}{lcl} \sigma_1^{-1} \circ \sigma_0^{-1} \circ \varphi & : & \begin{array}{ll} a & \mapsto abad \\ b & \mapsto abb \\ c & \mapsto acd \\ d & \mapsto ad \end{array} \end{array}$$

which is again A_N -positive.

The algorithm cycles another six times. One may check that we obtain

$$\begin{array}{lll} \sigma_2 : & \begin{array}{l} a \mapsto a \\ b \mapsto b \\ c \mapsto cd \\ d \mapsto d \end{array} & \sigma_3 : \begin{array}{l} a \mapsto a \\ b \mapsto b \\ c \mapsto c \\ d \mapsto ad \end{array} & \sigma_4 : \begin{array}{l} a \mapsto a \\ b \mapsto b \\ c \mapsto ac \\ d \mapsto d \end{array} \\ \sigma_5 : & \begin{array}{l} a \mapsto ab \\ b \mapsto b \\ c \mapsto c \\ d \mapsto d \end{array} & \sigma_6 : \begin{array}{l} a \mapsto a \\ b \mapsto ab \\ c \mapsto c \\ d \mapsto d \end{array} & \sigma_7 : \begin{array}{l} a \mapsto ad \\ b \mapsto b \\ c \mapsto c \\ d \mapsto d \end{array} \end{array}$$

and $\varphi = \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \sigma_4 \circ \sigma_5 \circ \sigma_6 \circ \sigma_7$. Define the incidence matrix

$$M = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

of ψ , its dominant eigenvalue η and choose a positive eigenvector λ associated to η . The CIET $\delta = (\pi, \lambda)$ satisfies $\Sigma_\delta = \Sigma_\psi$, and φ is the δ -automorphism.

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